THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4230 2024-25
Lecture 10
February 12, 2025 (Wednesday)

1 Recall

Yesterday, we mentioned the following example:

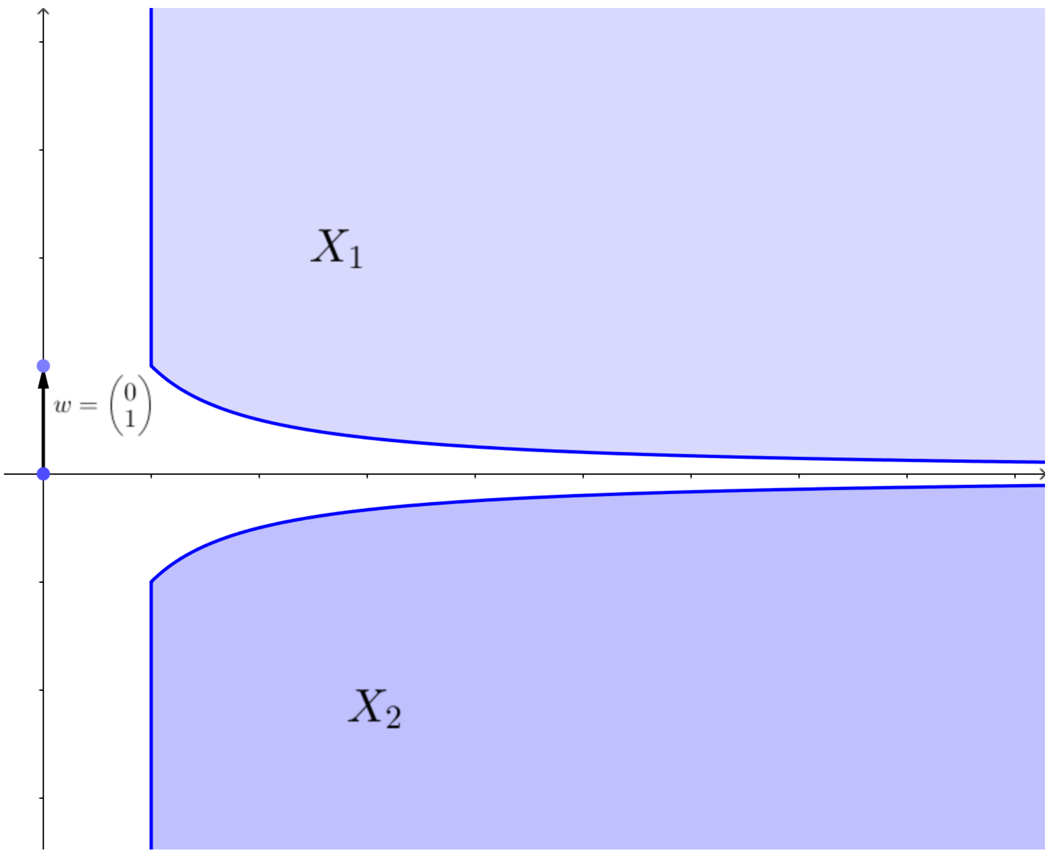
Example 1. If X_2 is not bounded, then the inequality may not be strict. See the following counterexample. Consider the following two sets:

$$X_1 := \left\{(x, y) : x \geq 1,\ y \leq -\frac{1}{x}\right\}$$

$$X_2 := \left\{(x, y) : x \geq 1,\ y \geq \frac{1}{x}\right\}$$

If we choose $w = (0, 1)^T$, then

$$\sup_{x_1 \in X_1} w^T x_1 = 0 = \inf_{x_2 \in X_2} w^T x_2$$



 Prepared by Max Shung

Example 2. Consider
$$X_1 = \{(x, 0) : x \in [0, 1]\}$$
$$X_2 = \{(x, 0) : x \in [0, 2]\}$$
Then $X_1 \subseteq X_2$, so we cannot separate these two convex sets. If we choose $w = (0, 1)$, then we still have
$$\sup_{x_1 \in X_1} w^T x_1 = 0 = \inf_{x_2 \in X_2} w^T x_2.$$

Remarks. From the above two examples, we can see
$$\sup_{x_1 \in X_1} w^T x_1 < \inf_{x_2 \in X_2} w^T x_2 \quad \text{and} \quad \sup_{x_1 \in X_1} w^T x_1 \leq \inf_{x_2 \in X_2} w^T x_2$$
are not good notions for **separation**.

2 Separation

Definition 1. X_1 and X_2 are **properly separated** by the linear form $w^T x$ if
$$\sup_{x_1 \in X_1} w^T x_1 \leq \inf_{x_2 \in X_2} w^T x_2$$
and
$$\inf_{x_1 \in X_1} w^T x_1 < \sup_{x_2 \in X_2} w^T x_2.$$

Proposition 1. *Let $X \subseteq \mathbb{R}^n$ be a non-empty convex set (not necessarily closed) and $y \notin X$. Then the following are equivalence:*

- *there exist $w \neq 0$ such that $w^T x$ separates properly X and $\{y\}$*
- $\sup_{x \in X} w^T x \leq w^T y$ *and* $\inf_{x \in X} w^T x < w^T y$.

Proof. 1. Assume that $y = 0$, then $\mathrm{Lin}(X) = \mathbb{R}^n$.
Let $\{x_i\}_{i \geq 1}$ is a dense subset of X.
For each $n \geq 1$, we consider the set
$$\mathrm{Conv}\left(\{x_i,\ 1 \leq i \leq n\}\right)$$
is convex and closed, and $y = 0 \notin \mathrm{Conv}\left(\{x_i,\ 1 \leq i \leq n\}\right) \subseteq X$. By the previous theorem, then there exists $w_n \neq 0$ such that
$$0 = w_n^T y > \max_{1 \leq i \leq n} w_n^T x_i \implies \frac{w_n^T y}{\|w_n\|} > \max_{1 \leq i \leq n} \frac{w_n^T x_i}{\|w_n\|}$$
Without loss of generality, we assume that $\|w_n\| = 1$.
Then there exists n_k such that $w_{n_k} \to w \neq 0$.
Taking limit $n_k \to \infty$, we have $0 \geq \sup_{i \geq 1} w^T x_i$.
Therefore, we have
$$\sup_{x \in X} w^T x = \sup_{i \geq 1} w^T x_i \leq 0 = w^T y$$
because $\{x_i\}_{i=1,2,\ldots}$ is dense in X.

2. If $\inf_{x \in X} w^T x = 0$, then it implies that $w^T x = 0, \quad \forall x \in X$, that is $0 \neq w \perp \mathrm{Lin}(X) = \mathbb{R}^n$.
This is a contradiction.

3. Now, we discuss the additional condition. If $y \neq 0$, we consider the *shifting set*

$$\tilde{X} := \{x - y : x \in X\}$$

Then y is separated with $X \iff 0$ is separated with $\tilde{X}$.

4. If $\mathrm{Lin}(X) \neq \mathbb{R}^n$, then we have the following cases:

   - **Case 1:** $y \in \mathrm{Lin}(X)$
     We replace $\mathbb{R}^n$ by $\mathrm{Lin}(X)$.
   - **Case 2:** $y \notin \mathrm{Lin}(X)$
     it is easy to separate $\{y\}$ and a linear subspace.

□

Proposition 2. *Let X_1 and X_2 be convex sets (not necessarily closed) such that $X_1 \cap X_2 = \emptyset$. Then X_1 and X_2 can be properly separated.*

Proof. Let $X = X_1 - X_2 = \{x = x_1 - x_2 : x_1 \in X_1,\, x_2 \in X_2\}$.
Then X is convex and $0 \notin X$. By the previous proposition, there exist $w \neq 0$ such that

$$\sup_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} w^T(x_1 - x_2) \leq w^T 0 = 0$$

$$\inf_{\substack{x_1 \in X_1 \\ x_2 \in X_2}} w^T(x_1 - x_2) < 0$$

This implies that X_1 and X_2 are properly separated by $w^T x$. □

Theorem 3. *Let X_1, X_2 be non-empty convex such that*

$$\mathrm{ri}(X_1) \cap \mathrm{ri}(X_2) = \emptyset$$

Then X_1 and X_2 can be properly separated.

Proof. Let $\widetilde{X_1} = \mathrm{ri}(X_1), \widetilde{X_2} = \mathrm{ri}(X_2)$.
Then $\widetilde{X_1}$ is dense in X_1 and $\widetilde{X_2}$ is dense in X_2, and $\widetilde{X_1} \cap \widetilde{X_2} = \emptyset$.
By the previous theorem, there exist $w \neq 0$ such that

$$\sup_{x_1 \in X_1} w^T x_1 = \sup_{x_1 \in \widetilde{X_1}} w^T x_1 \leq \inf_{x_2 \in \widetilde{X_2}} w^T x_2 = \inf_{x_2 \in X_2} w^T x_2$$

and

$$\inf_{x \in X_1} w^T x_1 = \inf_{x_1 \in \widetilde{X_1}} w^T x_1 < \sup_{x_2 \in \widetilde{X_2}} w^T x_2 = \sup_{x_2 \in X_2} w^T x_2$$

□

Remarks. The above theorem is an "if and only if" statement. If X_1 and X_2 can be properly separated, then $\mathrm{ri}(X_1) \cap \mathrm{ri}(X_2) = \emptyset$. We will prove the other direction next lesson.

— End of Lecture 10 —